Example 1:

Solve

$$y' = e^{x+y}.$$

Answer

We have from the ODE that

$$e^{-y}dy = e^{x}dx.$$

The integrand of the left side is a function of y only, while that of the right side is a function of x only. Such an equation is called separable, and is solved by one integration. We get

$$-e^{-y} = e^{x} + c.$$

On the other hand, a second-order ODE like (1.10), which has non-constant coefficients (the coefficient of y in (1.10) is not a constant) is much more difficult to solve, even though it is linear.

While it is usually difficult to obtain closed form solutions of linear equations like (1.10), some general properties of its solution can be deduced.

Consider the n^{th}-order ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0 y = b(x), \tag{1.14}$$

which is linear and inhomogeneous. Let Y be the solution of (1.14) with $b = 0$:

$$Y^{(n)} + a_{n-1}(x)Y^{(n-1)} + \cdots + a_0 Y = 0. \tag{1.15}$$

It is easy to show that if Y_1 is a solution of (1.15), so is cY_1. This can be proved by multiplying (1.15) by c. More generally, it is easy to prove that if Y_1, $Y_2 \cdots Y_n$ are solutions of (1.15), so is

$$c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n. \tag{1.16}$$

Therefore, once we find n independent solutions of (1.15), we may construct a solution of (1.15) with n arbitrary constants, which is the most general solution of (1.15).

It is also easy to prove that if y_P is a solution of (1.14), i.e.,

$$y_P^{(n)} + a_{n-1}(x)y_P^{(n-1)} + \cdots + a_0 y_P = b(x),$$

and if y satisfies (1.14), then $(y - y_P)$ satisfies (1.15). Thus we have

$$y = y_P + Y. \tag{1.17}$$

This tells us that an efficient way to find the most general solution of (1.14) is to do it in two steps:

Step 1: We first set $b = 0$ and find the most general solution of (1.15).

Step 2: We find just one solution (any solution) of (1.14), which we call y_P.

The most general solution of (1.14) is (1.17).

Note that there is another class of ODE which can be solved in closed forms. These equations are linear ODEs with constant coefficients. They can be either homogeneous or inhomogeneous.

Problem for the Reader:

Find a particular solution of the following differential equation:

$$(D^{100} + 1)y = e^x.$$

Answer

There is more than one way to find a particular solution of this equation, but we advocate doing it in the following way. We treat D as if it were a number and get

$$y_P = \frac{1}{D^{100} + 1} e^x,$$

where y_P is a particular solution of the differential equation. By using (1.2) with $m = 1$, we get

$$y_P = \frac{1}{1^{100} + 1} e^x = \frac{1}{2} e^x.$$

You may worry that since $(1 + D^2)^{-1}$ is not a polynomial of D, (1.2) does not apply. So let us try to understand what we've done.

Problem for the Reader:

Verify that the solution given above satisfies the differential equation.

Answer

$$(D^{100}+1)\frac{e^x}{2} = (1+1)\frac{e^x}{2} = e^x.$$

We mention that the conventional method of finding a particular solution of a differential equation of this kind is that of undetermined coefficients, which is based on an ansatz. For the present problem, the ansatz of the method of undetermined coefficient is that $y_p(x)$ is of the form ce^x . The coefficient c is determined by plugging this solution into the differential equation and requiring that the equation be satisfied. While it is not being stated explicitly, the method we advocate here also employs the same ansatz. Indeed, it is clear that, for the problem we just treated, our method here is just a shorthand version of the method of undetermined coefficients.

Problem for the Reader:

Find a particular solution of the following differential equation:

$$y'' - y = e^x.$$

Answer

A particular solution of the differential equation above is

$$y_P = \frac{1}{D^2-1}e^x.$$

If we apply (1.2), then the denominator in the above expression is zero and we get $y_P = \infty$, not a meaningful result.

Instead, we first write

$$y_P = \frac{1}{(D-1)}\frac{1}{(D+1)}e^x.$$

By applying (1.5), we get

$$y_P = e^x \frac{1}{D}\frac{1}{(D+2)}1. \tag{1.18}$$

By setting

$$\frac{1}{(D+2)}1 = \frac{1}{2},$$

(1.18) becomes

$$y_P = e^x \frac{1}{D}\frac{1}{2}.$$

Since the inverse of differentiation is integration, we assert that

$$\frac{1}{D}1 = \int dx = x + c.$$

This assertion can be proved by calling

$$\frac{1}{D}1 \equiv y.$$

We have

$$Dy = 1,$$

which tells us that y is equal to $x + c$. We may set c to zero as all we need is one particular solution. Then we have

$$y_P = \frac{e^x x}{2}.$$

Some readers may feel uneasy about applying (1.5) with $(D^2 - 1)^{-1}$, which is not a polynomial. So let us justify it. The particular solution $y_p(x)$ we seek satisfies

$$(D-1)(D+1)y_p(x) = e^x.$$

Let

$$y_p(x) \equiv e^x Y(x)$$

and substitute it into the equation for $y_p(x)$. Since $(D-1)(D+1)$ is a polynomial of D, it is justified to apply (1.5) and we get

$$D(D+2)Y(x) = 1,$$

which leads to (1.18).

Example 2:

Find the general solution of

$$y^{(100)} + y = \cosh x.$$

Answer

Step 1: We solve the corresponding homogeneous equation

$$Y^{(100)} + Y = 0.$$

Try

$$Y = e^{mx},$$

then m must satisfy

$$m^{100} = -1. \tag{1.19}$$

Equation (1.19) is a polynomial equation which has one hundred roots, which we will denote as m_n, $n = 1, \cdots 100$, with

$$m_n = e^{\dfrac{\pi(2n+1)i}{100}}.$$

Thus the complementary solution of the present example is

$$Y = c_1 e^{m_1 x} + \cdots + c_{100} e^{m_{100} x}. \tag{1.20}$$

Step 2: We find one particular solution of the ODE. We have

$$y_P = \frac{1}{D^{100} + 1} \cosh x = \frac{1}{D^{100} + 1} \frac{1}{2}(e^x + e^{-x}).$$

Making use of (1.2), we have

$$y_P = \frac{1}{1+1} \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2} \cosh x. \tag{1.21}$$

The most general solution is

$$y = y_P + Y. \tag{1.22}$$

We note that we first split $\cosh x$ into the sum of two exponentials, then it turned out that we made the same replacement $D^{100} \to 1$ for both exponentials. Thus we were wasting our time doing the splitting. Why is it so?

Let us note that

$$D \cosh mx = m \sinh mx.$$

In other words, D operating on $\cosh mx$ is not equal to a constant times $\cosh mx$. Thus it is true that the replacement (1.2) cannot be made when D operates on $\cosh mx$. However, by differentiating the equation above one more time, we get

$$D^2 \cosh mx = m^2 \cosh mx.$$

This means that we may make the replacement

$$D^2 \to m^2 \tag{1.23}$$

when D^2 operates on $\cosh mx$. With this understanding, y_P in Example 2 can be calculated without the need of splitting $\cosh x$ into the sum of two exponentials.

It's easy to prove that we may also make the replacement (1.23) if D^2 operates on $\sinh mx$.

Also note that, when applied to $\cos mx$, we may make the replacement

$$D^2 \to -m^2.$$

Example 3:

Find a particular solution for

$$y'' - y = \cosh x.$$

Answer

We have

$$y_p(x) = \frac{1}{D^2 - 1} \cosh x.$$

If we replace D^2 by unity, the right side blows up. Thus we write

$$y_p(x) = \lim_{m \to 1} \frac{1}{D^2 - 1} \cosh mx = \lim_{m \to 1} \frac{\cosh mx}{m^2 - 1}.$$

The solution $y_p(x)$ is still the same and therefore still blows up. We shall add a solution of the corresponding homogeneous equation to make it finite. Thus we try the particular solution

$$y_p(x) = \lim_{m \to 1} \frac{\cosh mx - \cosh x}{m^2 - 1}.$$

This is permissible as adding a solution of the homogeneous equation to a particular solution just gives another particular solution. By using l'Hopital's rule, we get

$$y_p(x) = \frac{x \sinh x}{2}.$$

Example 4:

Find the general solution of

$$y'' + 2y' + y = 2\sinh 3x + x + 3e^{-x}.$$

Answer

Step 1: We find the complementary solution by solving

$$m^2 + 2m + 1 = 0,$$

which has the double root $m = -1$. Hence we have found only one of the independent complementary solutions.

To find the other independent complementary solution, let

$$Y \equiv e^{-x} v,$$

and substitute the expression above into the equation

$$(D+1)^2 Y = 0.$$

We get

$$D^2 v = 0,$$

or

$$v = c_1 + c_2 x.$$

Thus the complementary solution is

$$Y = e^{-x}(c_1 + c_2 x).$$

A particular solution is

$$y_P = \frac{1}{D^2 + 2D + 1}(2\sinh 3x + x + 3e^{-x}).$$

Now

$$y_{P1} \equiv \frac{1}{D^2 + 2D + 1} 2\sinh 3x = \frac{1}{D+5}\sinh 3x,$$

where (1.23) has been used. We reduce the above equation further by multiplying the numerator and the denominator by $(D-5)$. Thus we have

$$y_{P1} = \frac{D-5}{D^2-25}\sinh 3x = \frac{D-5}{9-25}\sinh 3x = -\frac{3\cosh 3x - 5\sinh 3x}{16}.$$

Next we have

$$y_{P2} \equiv \frac{1}{(1+D)^2}x = (1-2D+\cdots)x = x-2,$$

as

$$D^2x = D^3x = \cdots = 0.$$

Next we have

$$y_{P3} \equiv \frac{1}{(D+1)^2}(3e^{-x}) = 3e^{-x}\frac{1}{D^2}1.$$

But the inverse of D is integration. Thus we have

$$y_{P3} = 3e^{-x}\int_0^x dx' \int_0^{x'} dx'' = \frac{3}{2}e^{-x}x^2.$$

The most general solution of the ODE in this example is the sum of y_{P1}, y_{P2}, y_{P3} and Y.

Below is a formula which will enable us to handle the general case in which the inhomogeneous term e^{mx} is a solution of the corresponding homogeneous equation. Let

$$P(D)y_P = e^{mx},$$

where

$$P(D) \equiv (D-m)^n p(D),$$

where

$$p(m) \neq 0.$$

We have

$$\begin{aligned} y_P(x) &= \frac{1}{(D-m)^n p(D)}e^{mx} = \frac{1}{(D-m)^n p(D)}e^{mx} \\ &= \frac{1}{(D-m)^n p(m)}e^{mx} \\ &= \frac{e^{mx}}{p(m)}\frac{1}{D^n}1 = \frac{e^{mx}}{p(m)}\frac{x^n}{n!}. \end{aligned}$$

Example 5:

Solve

$$y'' + y = f(x),$$

where $f(x)$ is an awful-looking function.

Answer

The complementary solution is

$$Y = a\cos x + b\sin x.$$

A particular solution is

$$y_P = \frac{1}{D^2+1} f(x) = \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right) f(x).$$

We have

$$\frac{1}{D-i} f(x) = \frac{1}{D-i} e^{ix}(e^{-ix} f(x)) = e^{ix}\frac{1}{D} e^{-ix} f(x) = e^{ix}\int_0^x dx' e^{-ix'} f(x'),$$

where (1.6) has been used.

Similarly

$$\frac{1}{D+i} f(x) = e^{-ix}\int_0^x dx' e^{ix'} f(x').$$

Thus

$$y_P = \int_0^x dx' \frac{e^{i(x-x')} - e^{-i(x-x')}}{2i} f(x') = \int_0^x dx' \sin(x-x') f(x').$$

Example 6:

Solve the boundary-value problem

$$\frac{d^2y}{dx^2} - y = g(x), \; -\infty < x < \infty,$$

with the boundary conditions

$$y(\infty) = y(-\infty) = 0.$$

It is assumed that the function $g(x)$ vanishes at infinity and has a Fourier transform.

Answer

Let us express $g(x)$ by

$$g(x) = \int_{-\infty}^{\infty} \tilde{g}(k)e^{ikx}\frac{dk}{2\pi},$$

where $\tilde{g}(k)$ is the Fourier transform of $g(x)$. Then a particular solution is

$$y_p(x) = \frac{1}{D^2 - 1}\int_{-\infty}^{\infty} \tilde{g}(k)e^{ikx}\frac{dk}{2\pi} = -\int_{-\infty}^{\infty} \frac{\tilde{g}(k)}{k^2+1}e^{ikx}\frac{dk}{2\pi}.$$

To verify that $y_P(x)$ above does vanish at infinity, we note that, when $|x| \gg 1$, the factor e^{ikx} in the integrand oscillates rapidly, resulting in great cancellations. Indeed, the Riemann-Lebesgue lemma asserts that $y_p(x)$ vanishes as $|x| \to \infty$. (We will discuss this more extensively in Chapter 7.)

The complementary solution is

$$ae^x + be^{-x},$$

Since e^x (e^{-x}) blows up as $x \to \infty$ ($x \to -\infty$), the boundary conditions at $x = \pm\infty$ is satisfied only if both a and b are zero.

Next we consider the differential equation

$$x^n\frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = 0.$$

The coefficient of the m^{th} derivative of y in the ODE is equal to the constant a_m times x^m, and is not a constant. This equation is unchanged if we replace x by cx, thus it is called equidimensional. This ODE can be solved in a closed form by the change of variable

$$X \equiv \ln x, \text{ or } x = e^X.$$

Problem for the Reader:

Express $x\dfrac{d}{dx}$ in terms of the variable X and D, here $D \equiv \dfrac{d}{dX}$. Also, do the same for $x^n\dfrac{d}{dx^n}$.

Answer

We have

$$\frac{d}{dx} = \left(\frac{dx}{dX}\right)^{-1} \frac{d}{dX} = e^{-X}\frac{d}{dX} = e^{-X}D,$$

Thus

$$x\frac{d}{dx} = D.$$

We also have

$$\frac{d^2}{dx^2} = (e^{-X}D)(e^{-X}D) = e^{-2X}(D-1)D,$$

thus

$$x^2\frac{d^2}{dx^2} = D(D-1).$$

More generally, we have

$$\frac{d^n}{dx^n} = e^{-nX}D(D-1)\cdot\cdot(D-n+1).$$

Thus

$$x^n\frac{d^n}{dx^n} = D(D-1)\cdot\cdot(D-n+1),\ n = 1, 2, 3\cdots. \tag{1.24}$$

Therefore, a homogeneous and equidimensional ODE can be transformed into an ODE with constant coefficients and is thus exactly solvable. The solutions are of the form $e^{mX} = x^m$. Indeed, when it operates on x^m, $x^n\dfrac{d^n}{dx^n}$ can be replaced by

$$m(m-1)\cdot\cdot(m-n+1).$$

We shall next consider a non-linear ODE.

Example 7:

Solve

$$y' = a(x) + b(x)y + c(x)y^2.$$

Note that the right side of the equation above is quadratic, not linear, in y. If $c(x) = 0$, the equation above is linear and is exactly solvable, as we have discussed. If $a(x) = 0$, this equation can be linearized by the change of the dependent variable $u = y^{-1}$. (See homework problem 1c). If $a(x)c(x) \neq 0$, this is a first-order non-linear equation known as Riccati's equation.