

Because $H_{0,t}$ commutes with I_1 ,

$$\begin{aligned} \ddot{r}(0,t) &= \sqrt{-1} \left(\int_X \operatorname{tr}(\ddot{F}_0 \cdot I_1) \wedge \omega^{n-1} \right. \\ &\quad - 2 \int_X \operatorname{tr}([\dot{A}_0, [\dot{A}_0^*, H_{0,t}] H_{0,t}^{-1}] \cdot I_1) \wedge \omega^{n-1} \\ &\quad - 2 \int_X \operatorname{tr}([\dot{A}_0, D'_0 \dot{H}_{0,t} \cdot H_{0,t}^{-1}] \cdot I_1) \wedge \omega^{n-1} \\ &\quad \left. + \int_X \operatorname{tr}(D''_0 \dot{\varphi}_{0,t} \cdot I_1) \wedge \omega^{n-1} \right). \end{aligned}$$

To analyze the sign of the above integration, we use the splitting $E = E_1 \oplus E_2$ to express

$$\dot{A}_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Because

$$H_{0,t} = \begin{pmatrix} \exp(\frac{t}{r_2}) \cdot I_1 & 0 \\ 0 & \exp(\frac{-t}{r_1}) \cdot I_2 \end{pmatrix},$$

the second term

$$-2\sqrt{-1} \int_X \operatorname{tr}([\dot{A}_0, [\dot{A}_0^*, H_{0,t}] H_{0,t}^{-1}] \cdot I_1) \wedge \omega^{n-1}$$

in $\ddot{r}(0,t)$ is, for $\alpha = \frac{1}{n_1} + \frac{1}{n_2}$,

$$\begin{aligned} -2\sqrt{-1}(1 - e^{-\alpha t}) \int_X \operatorname{tr}(C_{12} \wedge C_{12}^*) \wedge \omega^{n-1} \\ - 2\sqrt{-1}(1 - e^{\alpha t}) \int_X \operatorname{tr}(C_{21}^* \wedge C_{21}) \wedge \omega^{n-1}. \end{aligned}$$

Similarly, because of (2.4) and $F_s^{2,0} = F_s^{0,2} = 0$,

$$\begin{aligned} \sqrt{-1} \int_X \operatorname{tr}(\ddot{F}_0 \cdot I_1) \wedge \omega^{n-1} &= 2\sqrt{-1} \int_X \operatorname{tr}(C_{12} \wedge C_{12}^*) \wedge \omega^{n-1} \\ &\quad + 2\sqrt{-1} \int_X \operatorname{tr}(C_{21}^* \wedge C_{21}) \wedge \omega^{n-1}. \end{aligned}$$

The last term in $\ddot{r}(0,t)$ is zero because of Lemma 2.3; the next-to-last term is

$$\begin{aligned} -2\sqrt{-1} \int_X \operatorname{tr}(\dot{A}_0 \cdot D'_0 \dot{H}_{0,t} \cdot H_{0,t}^{-1} \cdot I_1) \wedge \omega^{n-1} \\ + 2\sqrt{-1} \int_X \operatorname{tr}(D'_0 \dot{H}_{0,t} \cdot H_{0,t}^{-1} \cdot \dot{A}_0 \cdot I_1) \wedge \omega^{n-1}, \end{aligned}$$

which vanishes because $D_0''^* \dot{A}_0 = 0$ and Lemma 2.3. Therefore,

$$\begin{aligned} \ddot{r}(0, t) &= \sqrt{-1} e^{-\alpha t} \int_X \operatorname{tr}(C_{12} \wedge C_{12}^*) \wedge \omega^{n-1} \\ &\quad + \sqrt{-1} e^{\alpha t} \int_X \operatorname{tr}(C_{21}^* \wedge C_{21}) \wedge \omega^{n-1}. \end{aligned}$$

Because the associated cohomology class $[C_{ij}] = \kappa_{ij}$ and κ_{21} and κ_{12} are both non-zero,

$$\begin{aligned} A &= \sqrt{-1} \int_X \operatorname{tr}(C_{12} \wedge C_{12}^*) \wedge \omega^{n-1} \quad \text{and} \\ B &= -\sqrt{-1} \int_X \operatorname{tr}(C_{21}^* \wedge C_{21}) \wedge \omega^{n-1} \end{aligned}$$

are positive. Hence, for sufficiently small s , the value $r(s, t) > 0$ for $t < \frac{1}{2\alpha} \ln \frac{A}{B}$ and $r(s, t) > 0$ for $t > \frac{1}{2\alpha} \ln \frac{A}{B}$. Hence, there is a function $t = \rho(s)$ so that $r(s, \rho(s)) = 0$. This proves that the system $L_s(H) = 0$ is solvable for small s . Here, the function $\rho(s)$ is not necessarily continuous, but $\lim_{s \rightarrow 0} \rho(s) = \frac{1}{2\alpha} \ln \frac{A}{B}$.

3. Linearization of Strominger's system

In this section, we will study the linearization of Strominger's system. Before we do this, we will first rephrase the system (1.1)–(1.4) in the form that is easier to handle.

We fix a Calabi–Yau threefold (X, ω_0) and a $(3, 0)$ -holomorphic form Ω so that $\Omega \wedge \bar{\Omega} = \omega_0^3$. We let (E, D'') be a rank r holomorphic bundle over X such that $c_1(E) = 0$ and $c_2(E) = c_2(X)$. We then choose a hermitian metric H on E and let $D_H = D'_H \oplus D''$ be the hermitian connection of (E, D'', H) ; its curvature $F_H = D_H \circ D_H$ satisfies

$$F_H^{2,0} = F_H^{0,2} = 0.$$

Thus, the second equation of the Strominger's system follows automatically.

The fourth equation of the system is a non-linear equation of a hermitian form ω involving the adjoint d_ω^* of ω . It turns out that this equation is equivalent to

$$d(\|\Omega\|_\omega \omega^2) = 0.$$

We now prove this equivalence. We let $\mathcal{H}(X)$ and $\mathcal{K}(X)$ be the cones of positive definite hermitian forms and Kahler forms on X respectively. Given an $\omega \in \mathcal{H}(X)$, we let $*_\omega$ be the (hermitian) star operator of ω ; and let d_ω^* be the adjoint of d with respect to the metric ω .

The hermitian star operator has an explicit local expression. Given a hermitian form ω on X , it induces canonical hermitian metrics on $T_{X,\mathbb{C}}$ and on $\wedge^k T_{X,\mathbb{C}}^\vee$. Let $(\cdot, \cdot)_\omega$ be the hermitian metric on $\wedge^k T_{X,\mathbb{C}}^\vee$ and $\frac{1}{3!}\omega^3$ its associated volume form on X . The star operator $*_\omega$ is the \mathbb{C} -linear operator defined via

$$(\varphi, \psi)_\omega \cdot \frac{\omega^3}{3!} = \varphi \wedge *_\omega \bar{\psi}.$$

Let $p \in X$ be any point and let $\varphi_1, \varphi_2, \varphi_3$ be an $(\cdot, \cdot)_\omega$ -orthonormal basis (a moving frame) of the $(1,0)$ -forms near p obeying $(\varphi_i, \varphi_j)_\omega = 2\delta_{ij}$. Then, the hermitian form

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^3 \varphi_i \wedge \bar{\varphi}_i.$$

For any subset $I = \{i_1, \dots, i_k\} \subset \{1, 2, 3\}$, we denote by $\varphi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$, and denote by I° the complement $\{1, 2, 3\} - I$. Following this convention,

$$(3.1) \quad *_\omega(c \bar{\varphi}_I \wedge \varphi_J) = \epsilon_{IJ} \sqrt{-1} \ 2^{|I|+|J|-3} c \varphi_{I^\circ} \wedge \bar{\varphi}_{J^\circ}, \quad c \in \mathbb{C},$$

where ϵ_{IJ} is the parity of permuting $(I, J; I^\circ, J^\circ) \mapsto (1, 2, 3; 1', 2', 3')$.

We now re-state and prove the mentioned equivalence.

Lemma 3.1. *Let ω_0 be the reference Kahler form as before. Then, the equation (1.4) is equivalent to*

$$(3.2) \quad *_{\omega_0} d(\|\Omega\|_\omega \omega^2) = 0.$$

Proof. Let f be a positive real valued function, then

$$d(f\omega^2) = f d\omega^2 + df \wedge \omega^2 = 2fd *_\omega \omega + df \wedge \omega^2.$$

Thus,

$$*_\omega d(f\omega^2) = 2f *_\omega d *_\omega \omega + *_\omega(df \wedge \omega^2) = -2fd *_\omega^* \omega + 2d_c f.$$

Here, we have used the identity

$$*_\omega(df \wedge \omega^2) = 2d_c f,$$

which holds for all hermitian form ω . Replacing f by $\|\Omega\|$, we obtain

$$*_\omega d(\|\Omega\|_\omega \omega^2) = 2\|\Omega\|_\omega (-d_\omega^* \omega + d_c \log \|\Omega\|_\omega),$$

which vanishes if and only if

$$d_\omega^* \omega = d_c \log \|\Omega\|_\omega.$$

Finally, since $*_\omega$ and $*_{\omega_0}$ are both isomorphisms, $*_\omega d(\|\Omega\|_\omega^{-1} \omega^2) = 0$ if and only if

$$*_{\omega_0} d(\|\Omega\|_\omega \omega^2) = 0.$$

This proves the lemma. q.e.d.

To apply the implicit function theorem, we need to specify the range of the operators associated to Strominger's system. For that, noting that $2dd_c = \sqrt{-1}\partial\bar{\partial}$, we let $R(dd_c) \subset \Omega_{\mathbb{R}}^{2,2}(X)$ and $R(d_{\omega_0}^*) \subset \Omega_{\mathbb{R}}^1(X)$ be the range of

$$dd_c : \Omega_{\mathbb{R}}^{1,1}(X) \rightarrow \Omega_{\mathbb{R}}^{2,2}(X) \quad \text{and} \quad d_{\omega_0}^* : \Omega_{\mathbb{R}}^{1,1}(X) \rightarrow \Omega_{\mathbb{R}}^1(X).$$

Because (X, ω_0) is a Kahler manifold, by $\partial\bar{\partial}$ -lemma, a real form $\alpha \in R(dd_c)$ if and only if $d\alpha = 0$. Hence, after picking a usual Banach norm on $\Omega_{\mathbb{R}}^{2,2}(X)$, $R(dd_c)$ is closed in it. As to $R(d_{\omega_0}^*)$, since $d_{\omega_0}^*$ is part of an elliptic complex, it is also closed. This way, after replacing (1.4) by (3.2) and omitting the equation (1.2), the Strominger's system is equivalent to the vanishing of the operator

$$(3.3) \quad \mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2 \oplus \mathbf{L}_3 : \mathcal{H}(E)_1 \times \mathcal{H}(X) \longrightarrow \Omega_{\mathbb{R}}^6(\mathfrak{su}E) \oplus R(dd_c) \oplus R(d_{\omega_0}^*),$$

defined by

$$(3.4) \quad \mathbf{L}_1(H, \omega) = H^{-1/2} F_H H^{1/2} \wedge \omega^2 \in \Omega_{\mathbb{R}}^6(\mathfrak{su}E);$$

$$(3.5) \quad \mathbf{L}_2(H, \omega) = \frac{1}{2} dd_c \omega + (\text{tr}(F_H \wedge F_H) - \text{tr}(R_\omega \wedge R_\omega)) \in \Omega_{\mathbb{R}}^{2,2}(X);$$

$$(3.6) \quad \mathbf{L}_3(H, \omega) = *_{\omega_0} d(\|\Omega\|_{\omega} \omega^2) \in \Omega_{\mathbb{R}}^1(X).$$

Because $c_2(E) = c_2(TX)$ and X is a Kahler manifold, by $\partial\bar{\partial}$ -lemma the image of \mathbf{L}_2 lies in $R(P)$. As to \mathbf{L}_3 , because

$$*_{\omega_0} d = \pm *_{\omega_0} d *_{\omega_0} *_{\omega_0}^{-1} = \mp d_{\omega_0}^* *_{\omega_0}^{-1},$$

its image lies in the range of $d_{\omega_0}^*$ as well. Therefore, the operator \mathbf{L} is well-defined.

Proposition 3.2. *Suppose $\mathbf{L}(H, \omega_0) = 0$. Then, the three summands of the linearization of \mathbf{L} at (H, ω_0) are*

$$\delta\mathbf{L}_1(H, \omega_0)(\delta h, \delta\omega) = D'' D'_H \delta h \wedge \omega_0^2 + 2H^{-1/2} F_H H^{1/2} \wedge \omega_0 \wedge \delta\omega;$$

$$\begin{aligned} \delta\mathbf{L}_2(H, \omega_0)(\delta h, \delta\omega) &= \frac{1}{2} dd_c \delta\omega + 2(\text{tr}(\delta F_H(\delta h) \wedge F_H) \\ &\quad - \text{tr}(\delta R_{\omega_0}(\delta\omega) \wedge R_{\omega_0})); \end{aligned}$$

$$\delta\mathbf{L}_3(H, \omega_0)(\delta h, \delta\omega) = 2d_{\omega_0}^* \delta\omega - d_{\omega_0}^*((\delta\omega, \omega_0)_{\omega_0} \omega_0).$$

Here, as before, we follow the convention $\delta H = H^{-1/2} \delta h H^{-1/2}$.

Proof. The formula for $\delta\mathbf{L}_1$ is well-known [27]; the formula for $\delta\mathbf{L}_2$ in the written form is a tautology; we stop short of finding an explicit form of δR since the current form is sufficient for our purposes.

We now prove the formula for $\delta\mathbf{L}_3$. Let ω_t be a smooth variation of the hermitian form ω_0 ; let $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ be an orthonormal basis of $(1, 0)$ -forms, smooth in t , expressed in a holomorphic coordinate (z_1, z_2, z_3) near $p \in X$ by

$$\varphi_i(t) = \sum_j b_{ij}(t) dz_j, \quad b_{ij}(0)(p) = \delta_{ij} \quad \text{and} \quad (\varphi_i(t), \varphi_j(t))_{\omega_t} = 2\delta_{ij}.$$

We can compute explicitly $\frac{d}{dt}(\omega_t^2)|_{t=0}$. First,

$$\omega_t^2 = \frac{1}{2} \sum \varphi_{i^\circ}(t) \wedge \bar{\varphi}_{i^\circ}(t) = \frac{1}{2} \sum_{i,l,k} c_{ik}(t) \bar{c}_{il}(t) dz_{k^\circ} \wedge d\bar{z}_{l^\circ},$$

where $c_{ij}(t)$ is the ij -th minor of the matrix $(b_{ij}(t))_{3 \times 3}$; namely

$$(3.7) \quad (c_{ij}(t))^t = \det(b_{ij}(t)) \cdot (b_{ij}(t))^{-1}.$$

Hence, at p ,

$$\frac{d}{dt}\omega_t^2|_{t=0} = \frac{1}{2} \sum (\dot{c}_{lk}(0) + \dot{\bar{c}}_{kl}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ}.$$

Using the identity (3.7) above,

$$\dot{c}_{lk}(0) + \dot{\bar{c}}_{kl}(0) = -\dot{b}_{kl}(0) - \dot{\bar{b}}_{lk}(0) + c_{lk}(0) \sum \dot{b}_{ii}(0) + \bar{c}_{kl}(0) \sum_i \dot{\bar{b}}_{ii}(0).$$

Therefore, at p ,

$$\begin{aligned} \frac{d}{dt}\omega_t^2|_{t=0} &= \frac{-1}{2} \sum_{l,k} (\dot{b}_{kl}(0) + \dot{\bar{b}}_{lk}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ} \\ &\quad + \frac{1}{2} \left(\sum_k dz_{k^\circ} \wedge d\bar{z}_{k^\circ} \right) \cdot \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0) \right). \end{aligned}$$

On the other hand, $\omega_0^2 = \frac{1}{2} \sum dz_{k^\circ} \wedge d\bar{z}_{k^\circ}$. Hence,

$$(3.8) \quad \frac{d}{dt}\omega_t^2|_{t=0} = \frac{-1}{2} \sum_{l,k} (\dot{b}_{kl}(0) + \dot{\bar{b}}_{lk}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ} + \omega_0^2 \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0) \right).$$

Next, we compute

$$\frac{d}{dt} \log \|\Omega\|_{\omega_t}^2|_{t=0} = -\frac{d}{dt} \frac{\omega_t^3}{\omega_0^3}|_{t=0} = -\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0).$$

Adding $\|\Omega\|_{\omega_0} \equiv 1$, we get

$$\begin{aligned} \frac{d}{dt}(\|\Omega\|_{\omega_t} \omega_t^2)|_{t=0} &= \left(\frac{1}{2} \omega_0^2 \frac{d}{dt} \log \|\Omega\|_{\omega_t}^2 + \frac{d}{dt} \omega_t^2 \right) |_{t=0} \\ &= -\frac{1}{2} \sum_{l,k} (\dot{b}_{kl}(0) + \dot{b}_{lk}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ} \\ &\quad + \frac{1}{2} \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0) \right) \omega_0^2. \end{aligned}$$

On the other hand, at p

$$\frac{d}{dt} \omega_t |_{t=0} = \frac{\sqrt{-1}}{2} \sum_i \dot{\varphi}_i \wedge \bar{\varphi}_i + \varphi_i \wedge \dot{\bar{\varphi}}_i = \frac{\sqrt{-1}}{2} \sum_{i,j} (\dot{b}_{ji}(0) + \dot{\bar{b}}_{ij}(0)) dz_i \wedge d\bar{z}_j.$$

Hence,

$$*_\omega_0 \dot{\omega}_0 = \frac{1}{4} \sum_{i,j} (\dot{b}_{ji}(0) + \dot{\bar{b}}_{ij}(0)) dz_{i^\circ} \wedge d\bar{z}_{j^\circ}.$$

Combined, we obtain

$$\frac{d}{dt}(\|\Omega\|_{\omega_t} \omega_t^2)|_{t=0} = -2 *_\omega_0 \dot{\omega}_0 + \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0) \right) \frac{\omega_0^2}{2}.$$

It remains to treat the term $\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0)$. From

$$(\dot{\omega}_0, \omega_0)_{\omega_0} \frac{\omega_0^3}{3!} = \dot{\omega}_0 \wedge *_\omega_0 \omega_0 \quad \text{and} \quad *_\omega_0 \omega_0 = \frac{1}{4} \sum dz_{k^\circ} \wedge d\bar{z}_{k^\circ},$$

we get

$$\begin{aligned} &\dot{\omega}_0 \wedge *_\omega_0 \omega_0 \\ &= \frac{\sqrt{-1}}{8} \sum (\dot{b}_{ij}(0) + \dot{\bar{b}}_{ji}(0)) dz_i \wedge d\bar{z}_j \wedge dz_{k^\circ} \wedge d\bar{z}_{k^\circ} \\ &= -\frac{\sqrt{-1}}{8} \sum (\dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0)) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3; \end{aligned}$$

hence,

$$(\dot{\omega}_0, \omega_0)_{\omega_0} = \frac{\dot{\omega}_0 \wedge *_\omega_0 \omega_0}{\omega_0^3/3!} = \sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0).$$

This proves that

$$\begin{aligned} \frac{d}{dt}(\|\Omega\|_{\omega_t} \omega_t^2)|_{t=0} &= -2 *_\omega_0 \dot{\omega}_0 + \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0) \right) \omega_0^2 \\ &= -2 *_\omega_0 \dot{\omega}_0 + *_\omega_0 (\dot{\omega}_0, \omega_0)_{\omega_0} \omega_0. \end{aligned}$$

Finally, Applying $*_{\omega_0} d$ to both sides of this identity, we obtain

$$\frac{d}{dt} *_\omega_0 d(\|\Omega\|_{\omega_t} \omega_t^2)|_{t=0} = 2d^*_{\omega_0} \dot{\omega}_0 - d^*_{\omega_0} ((\dot{\omega}_0, \omega_0)_{\omega_0} \omega_0).$$

This proves the Proposition. q.e.d.

Strominger's system admits a class of reducible solutions. Let

$$(E, D_0'') = \mathbb{C}_X^{\oplus r} \oplus TX$$

be the direct sum of the trivial holomorphic bundle $\mathbb{C}_X^{\oplus r}$ and the tangent bundle TX . We fix an isomorphism $\wedge^{r+3}E \cong \mathbb{C}_X$; we endow E with the hermitian metric \langle, \rangle that is a direct sum of a constant metric on $\mathbb{C}_X^{\oplus r}$ and the Calabi–Yau metric ω_0 on TX . We normalize \langle, \rangle so that its induced metric on $\wedge^{r+3}E \cong \mathbb{C}_X$ is the constant one metric. As before, the metric \langle, \rangle is identified with the identity endomorphism $I: E \rightarrow E$.

Now, let $\mathcal{H}_{r \times r}^+$ be the space of positive definite hermitian symmetric $r \times r$ metrics; let I_1 and I_2 be the identity endomorphisms of $\mathbb{C}_X^{\oplus r}$ and TX respectively. By abuse of notation, for $T \in \mathcal{H}_{r \times r}^+$, we also view it as the constant endomorphism of $\mathbb{C}_X^{\oplus r}$ induced by T , viewed as an endomorphism of E . Then, the assignment

$$T \in \mathcal{H}_{r \times r}^+ \longmapsto H_T = T \oplus |T|^{-1/3} I_2 \in \mathcal{H}(E)_1, \quad |T| = \det T,$$

associates each $T \in \mathcal{H}_{r \times r}^+$ to a hermitian metric of E .

Obviously, the hermitian curvature F_{H_T} of $(E, \langle, \rangle_{H_T})$ is $0 \oplus R_{\omega_0}$; hence, $F_{H_T} \wedge F_{H_T} = R_{\omega_0} \wedge R_{\omega_0}$. Because ω_0 is d -closed,

$$\mathbf{L}_2(H_T, \omega_0) = \frac{1}{2} dd_c \omega_0 + \text{tr}(F_{H_T} \wedge F_{H_T}) - \text{tr}(R_{\omega_0} \wedge R_{\omega_0}) = 0.$$

Further, because \langle, \rangle_{H_T} is Hermitian–Yang–Mills, and because $d_{\omega_0}^* \omega_0 = 0$ and $\Omega \wedge \bar{\Omega} = \omega_0^3$, $\mathbf{L}_1(H_T, \omega_0) = \mathbf{L}_3(H_T, \omega_0) = 0$. Therefore, (H_T, ω_0) is a solution to $\mathbf{L}(H, \omega) = 0$. Indeed, for any constant $c > 0$, the pair $(H_T, c\omega_0)$ is a solution to $\mathbf{L} = 0$. These solutions are reducible because the vector bundle E splits under the hermitian connection D_{H_T} . In this paper, we will call such solutions the *trivial* solutions to Strominger's system.

To construct irreducible solutions to Strominger's system, we will first form a family of holomorphic structures D_s'' on E that is a smooth deformation of D_0'' ; we then show that certain trivial solutions to Strominger's system on (E, D_0'') can be extended to (irreducible) solutions on (E, D_s'') . We shall prove this by applying implicit function theorem to the operator \mathbf{L} of (3.3).

To this end, we pick an integer k and a large p and endow the domain and the target of \mathbf{L} the Banach space structures as indicated:

$$\mathcal{H}(E)_{1, L_k^p} \times \mathcal{H}(X)_{L_k^p} \longrightarrow \Omega_{\mathbb{R}}^6(\mathfrak{su}E)_{L_{k-2}^p} \oplus R(dd_c)_{L_{k-2}^p} \oplus R(d_{\omega_0}^*)_{L_{k-1}^p}.$$

\mathbf{L} becomes a smooth operator and its linearized operator $\delta\mathbf{L}$ at a solution (H, ω) becomes a linear map

$$\Omega^0(\mathfrak{H}\mathfrak{e}\mathfrak{r}^0 E)_{L_k^p} \oplus \Omega_{\mathbb{R}}^{1,1}(X)_{L_k^p} \longrightarrow \Omega_{\mathbb{R}}^6(\mathfrak{su}E)_{L_{k-2}^p} \oplus R(dd_c)_{L_{k-2}^p} \oplus R(d_{\omega_0}^*)_{L_{k-1}^p}.$$

Here, we used the canonical isomorphisms $T_H\mathcal{H}(E)_{1,L_k^p} \cong \Omega^0(\mathfrak{H}\mathfrak{e}\mathfrak{r}^0 E)_{L_k^p}$ and $T_\omega\mathcal{H}(X)_{L_k^p} \cong \Omega_{\mathbb{R}}^{1,1}(X)_{L_k^p}$. For simplicity, in the following, we will abbreviate

$$\mathcal{W}_1 = \Omega_{\mathbb{R}}^6(\mathfrak{su}E)_{L_{k-2}^p} \quad \text{and} \quad \mathcal{W}_2 = R(dd_c)_{L_{k-2}^p} \oplus R(d_{\omega_0}^*)_{L_{k-1}^p}.$$

Thus, $\delta\mathbf{L}(H, \omega)$ is a linear map

$$(3.9) \quad \delta\mathbf{L}(H, \omega) : \Omega^0(\mathfrak{H}\mathfrak{e}\mathfrak{r}^0 E)_{L_k^p} \oplus \Omega_{\mathbb{R}}^{1,1}(X)_{L_k^p} \longrightarrow \mathcal{W}_1 \oplus \mathcal{W}_2.$$

To study the kernel and the cokernel of $\delta\mathbf{L}$ at a trivial solution $(H_T, c\omega_0)$, we will first look at the linear map

$$(3.10) \quad \mathbf{F}(\delta h) = D_0'' D_{0,H_T}'(\delta h) \wedge \omega_0^2 : \Omega^0(\mathfrak{H}\mathfrak{e}\mathfrak{r}^0 E)_{L_k^p} \longrightarrow \Omega_{\mathbb{R}}^6(\mathfrak{su}E)_{L_{k-2}^p}.$$

Here, according to our convention, $D_{H_T} = D_{0,H_T}' \oplus D_0''$ is the hermitian connection of (E, D_0'', H_T) for a $T \in \mathcal{H}_{r \times r}^+$. Since $(E, D_0'') = \mathbb{C}^{\oplus r}_X \oplus TX$, the above is a linear elliptic operator of index 0 whose kernel is

$$V_0 = \{M \oplus aI_2 \mid M \in \text{End}(\mathbb{C}^{\oplus r}), M = M^*, \text{tr } M + 3a = 0\}$$

and cokernel is

$$(3.11) \quad V_1 = \omega_0^3 \cdot V_0 \subset \mathcal{W}_1 = \Omega_{\mathbb{R}}^6(\mathfrak{su}E)_{L_{k-2}^p}.$$

We let \mathbf{P} be the obvious projection

$$\mathbf{P} : \mathcal{W}_1 \longrightarrow \mathcal{W}_1/V_1.$$

Proposition 3.3. *Let (X, ω_0) , Ω , \langle, \rangle and $T \in \mathcal{H}_{r \times r}^+$ be as before. Then, there is a constant C so that for any $c > C$, the linear operator*

$$\mathbf{P} \circ \delta\mathcal{L}_1(H_T, c\omega_0) \oplus \delta\mathcal{L}_2(H_T, c\omega_0) \oplus \delta\mathcal{L}_3(H_T, c\omega_0)$$

from $\Omega^0(\mathfrak{H}\mathfrak{e}\mathfrak{r}^0 E)_{L_k^p} \oplus \Omega_{\mathbb{R}}^{1,1}(X)_{L_k^p}$ to $\mathcal{W}_1/V_1 \oplus \mathcal{W}_2$ is surjective.

Proof. As we shall see, the proof of the Proposition relies on the understanding of the operator

$$T : \Omega_{\mathbb{R}}^{1,1}(X)_{L_k^p} \longrightarrow \mathcal{W}_2$$

defined by, after setting $P = \frac{1}{2}dd_c = \sqrt{-1}\partial\bar{\partial}$,

$$T\mu = (P\mu, 2d_{\omega_0}^*\mu - d_{\omega_0}^*((\mu, \omega_0)_{\omega_0}\omega_0)).$$

Before we go on, we remark that since in the proof of this Proposition we will solely work with the Kahler form ω_0 , for convenience, we will abbreviate $*_{\omega_0}$ and $d_{\omega_0}^*$ to $*$ and d^* .

For the starter, we form the linear operator S :

$$S\mu = 2\mu - (\mu, \omega_0)_{\omega_0} \omega_0 : \Omega_{\mathbb{R}}^{1,1} \longrightarrow \Omega_{\mathbb{R}}^{1,1}$$

and its inverse

$$S^{-1}\phi = \frac{1}{2}(\phi - (\phi, \omega_0)_{\omega_0} \omega_0).$$

Then, by setting $\phi = S\mu$, $T\mu$ can be expressed as

$$T\mu = T \circ S^{-1}\phi = (P \circ S^{-1}\phi, d^*\phi).$$

Then, applying the Hodge decomposition to $\phi \in \Omega_{\mathbb{R}}^{1,1}(X)$,

$$\phi = dd^*\psi + d^*d\psi + h$$

for a real $(1,1)$ -form ψ and harmonic h . By the $\partial\bar{\partial}$ -lemma, we can rewrite $d^*d\psi = *P\alpha$ for a real form α .

As to the harmonic h , we check that the pairing $(h, \omega_0)_{\omega_0}$ is constant. Since (X, ω_0) is Kahler,

$$d_c * h = d^* * h \wedge \omega_0 - d^*(* h \wedge \omega_0);$$

and since $d^* * h = d_c * h = 0$, $d^*(* h \wedge \omega_0) = 0$. Hence, the defining identity

$$(3.12) \quad (h, \omega_0) * 1 = * h \wedge \omega_0$$

forces $(h, \omega_0)_{\omega_0}$ to be a constant. Therefore, the space of harmonic forms $\mathbb{H} \subset \Omega_{\mathbb{R}}^{1,1}(X)$ lies in the kernel of both T and $T \circ S^{-1}$.

With this said, to study the surjectivity of T , we only need to look at those ϕ that are orthogonal to \mathbb{H} under the L^2 -intersection pairing

$$\langle u, v \rangle = \int_X (u, v)_{\omega_0} * 1.$$

In particular, such ϕ has decomposition

$$\phi = *P\alpha + d^*d\psi,$$

and

$$T \circ S^{-1}\phi = (P \circ S^{-1}(*P\alpha) + P \circ S^{-1}(dd^*\psi), d^*d(d^*\psi)).$$

To proceed, we look at the operator U :

$$(3.13) \quad U\alpha = 2 * P \circ S^{-1}(*P\alpha) = *P(*P\alpha - (*P\alpha, \omega_0)_{\omega_0} \omega_0).$$

Because

$$P^* = (\sqrt{-1}\partial\bar{\partial})^* = -\sqrt{-1}\bar{\partial}^*\partial^* = *\sqrt{-1}\partial\bar{\partial}* = *P^*,$$

$U\alpha$ can be re-written as

$$(3.14) \quad U\alpha = P^*P\alpha - *P((*P(\alpha), \omega_0)_{\omega_0} \omega_0).$$

To proceed, we need to simplify the operator U . We first use the identities

(3.15)

$$\partial^* \mu \wedge \omega_0 - \partial^*(\mu \wedge \omega_0) = \sqrt{-1} \bar{\partial} \mu \quad \text{and} \quad \bar{\partial}^* \mu \wedge \omega_0 - \bar{\partial}^*(\mu \wedge \omega_0) = -\sqrt{-1} \partial \mu,$$

which hold for all Kahler manifolds, to derive

$$\partial^*(f \omega_0^2) = -2\sqrt{-1} \bar{\partial} f \wedge \omega_0.$$

Using $(*P\alpha, \omega_0)_{\omega_0} = *(P\alpha \wedge \omega_0)$, which follows from (3.12), we have

$$P((*P\alpha, \omega_0)_{\omega_0} \omega_0) = P(*(P\alpha \wedge \omega_0) \wedge \omega_0) = -\sqrt{-1} * \bar{\partial}^* \partial^*(P\alpha \wedge \omega_0) \wedge \omega_0.$$

Applying the identities (3.15) further, we obtain

$$\partial^*(P\alpha \wedge \omega_0) = \partial^* P\alpha \wedge \omega - \sqrt{-1} \bar{\partial} P\alpha = \partial^* P\alpha \wedge \omega$$

and

$$\bar{\partial}^* \partial^*(P\alpha \wedge \omega_0) = \bar{\partial}^*(\partial^* P\alpha \wedge \omega_0) = \bar{\partial}^* \partial^* P\alpha \wedge \omega_0 + \sqrt{-1} \partial \bar{\partial}^* P\alpha \wedge \omega_0.$$

Put together, we obtain

$$\begin{aligned} P((*P\alpha, \omega_0)_{\omega_0} \omega_0) &= -\sqrt{-1} * \bar{\partial}^* \partial^*(P\alpha \wedge \omega_0) \wedge \omega_0 \\ &= -\sqrt{-1} * (\bar{\partial}^* \partial^* P\alpha \wedge \omega_0 + \sqrt{-1} \partial \bar{\partial}^* P\alpha \wedge \omega_0) \wedge \omega_0 \\ &= *(P^* P\alpha \wedge \omega_0) \wedge \omega_0 + *(\partial \bar{\partial}^* P\alpha \wedge \omega_0) \wedge \omega_0. \end{aligned}$$

Because $\partial \bar{\partial}^* P\alpha = \square_{\partial} P\alpha$ since $\partial P\alpha = 0$, the operator U becomes

$$(3.16) \quad U(\alpha) = P^* P\alpha - *(*(P^* P\alpha \wedge \omega) \wedge \omega_0) - *(*(\square_{\partial} P\alpha \wedge \omega_0) \wedge \omega_0).$$

To continue, recall that for $\nu \in \Omega_{\mathbb{R}}^{1,1}(X)$ such that $(\nu, \omega_0)_{\omega_0} = 0$, $*(\nu \wedge \omega_0) = -\nu$. Hence,

$$\begin{aligned} *(\nu \wedge \omega_0) &= *(\nu - \frac{1}{3}(\nu, \omega_0)_{\omega_0} \omega_0) \wedge \omega_0 + \frac{1}{3} *((\nu, \omega_0)_{\omega_0} \omega_0^2) \\ &= -\nu + (\nu, \omega_0)_{\omega_0} \omega_0 \end{aligned}$$

and

$$*(*(\nu \wedge \omega_0) \wedge \omega_0) = *(-\nu \wedge \omega_0 + (\nu, \omega_0)_{\omega_0} * \omega_0^2) = \mu + (\nu, \omega_0)_{\omega_0} \omega_0.$$

Therefore, by (3.16),

$$\begin{aligned} U\alpha &= P^* P\alpha - (P^* P\alpha + (P^* P\alpha, \omega_0)_{\omega_0} \omega_0) - *(*(\square_{\partial} P\alpha \wedge \omega_0) \\ &= - *(*(\square_{\partial} P\alpha \wedge \omega_0) - (P^* P\alpha, \omega_0)_{\omega_0} \omega_0). \end{aligned}$$

Now, we are ready to derive the estimate that for a universal constant C (in the sense that it only depends on (X, ω_0)),

(3.17)

$$C^{-1} \|T \circ S^{-1} \phi\| \leq \|P\alpha\|_{L_k^p} + \|dd^* \psi\|_{L_k^p} \leq C \|T \circ S^{-1} \phi\|, \quad \forall \phi \perp \mathbb{H}.$$

First, note that the first inequality holds because $T \circ S^{-1}$ is a bounded operator. As to the second, because $d^*d(d^*\psi) = \square_{\partial}d^*\psi$ and that $d^*\psi$ is orthogonal to the harmonic forms, the elliptic estimate ensures that for a universal constant C_1 ,

$$\|d^*\psi\|_{L^p_{k+1}} \leq C_1 \|\square_{\partial}d^*\psi\|_{L^p_{k-1}} \leq C_1 \|T \circ S^{-1}\phi\|.$$

Then, because

$$P \circ S^{-1}(dd^*\psi) = -\frac{1}{2}P(dd^*\psi, \omega_0)_{\omega_0}$$

and because the right-hand side involves the third differentiation of $d^*\psi$,

$$\|P \circ S^{-1}(dd^*\psi)\|_{L^p_{k-2}} \leq C_2 \|d^*\psi\|_{L^p_{k+1}} \leq C_1 C_2 \|T \circ S^{-1}\phi\|$$

holds for a universal constant C_2 . On the other hand,

$$(3.18) \quad \frac{1}{2} * U\alpha = T \circ S^{-1}\phi + \frac{1}{2}P \circ S^{-1}(dd^*\psi, \omega_0) - d^*d(d^*\psi),$$

the previous estimates ensure that there is a universal constant C_3 so that

$$(3.19) \quad \|U\alpha\|_{L^p_{k-2}} \leq C_3 \|T \circ S^{-1}\phi\|.$$

Because

$$d * (*\square_{\partial}P\alpha \wedge \omega_0) = 0,$$

the formula of $U\alpha$ before (3.17) gives

$$(3.20) \quad d(P^*P\alpha, \omega_0)_{\omega_0} \wedge \omega_0 = d(U\alpha).$$

Combined with

$$\int_X (P^*P\alpha, \omega_0)_{\omega_0} * 1 = \int_X (P\alpha, P\omega_0)_{\omega_0} * 1 = 0,$$

and that wedging ω forms an isomorphism from $\Omega_{\mathbb{R}}^{1,1}(X)$ to $\Omega_{\mathbb{R}}^{2,2}(X)$ whose inverse has bounded norm, (3.20) and (3.19) implies that

$$\|(P^*P\alpha, \omega_0)_{\omega_0}\|_{L^p_{k-2}} \leq C_4 \|U\alpha\|_{L^p_{k-2}} \leq C_3 C_4 \|T \circ S^{-1}\phi\|.$$

Thus, for a universal constant C_5 ,

$$\|\square_{\partial}P\alpha\|_{L^p_{k-2}} \leq C_5 \|T \circ S^{-1}\phi\|.$$

Finally, because \square_{∂} is elliptic,

$$\|P\alpha\|_{L^p_k} \leq C_6 \|T \circ S^{-1}\phi\|.$$

This proves that the second inequality in (3.17) holds for a universal constant C .